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Trigonometric Fundamentals

Definitions of Trigonometric Functions in Terms of Right Triangles

Let S and T be two sets. A **function** (or **mapping** or **map**) f from S to T (written as $f : S \to T$) assigns to each $s \in S$ exactly one element $t \in T$ (written $f(s) = t$); t is the **image** of s. For $S' \subseteq S$, let $f(S')$ (the image of S') denote the set of images of $s \in S'$ under f. The set S is called the **domain** of f, and $f(S)$ is the **range** of f.

For an angle θ (Greek "theta") between $0°$ and $90°$, we define trigonometric functions to describe the size of the angle. Let rays OA and OB form angle θ (see Figure 1.1). Choose point P on ray OA. Let Q be the **foot** (that is, the bottom) of the perpendicular line segment from P to the ray OB. Then we define the sine (sin), cosine (cos), tangent (tan), cotangent (cot), cosecant (csc), and secant (sec) functions as follows, where $|PQ|$ denotes the length of the line segment PQ:

$$\begin{aligned} \sin\theta &= \frac{|PQ|}{|OP|}, & \csc\theta &= \frac{|OP|}{|PQ|}, \\ \cos\theta &= \frac{|OQ|}{|OP|}, & \sec\theta &= \frac{|OP|}{|OQ|}, \\ \tan\theta &= \frac{|PQ|}{|OQ|}, & \cot\theta &= \frac{|OQ|}{|PQ|}. \end{aligned}$$

First we need to show that these functions are well defined; that is, they only depends on the size of θ, but not the choice of P. Let P_1 be another point lying on ray OA, and let Q_1 be the foot of perpendicular from P_1 to ray OB. (By the way, "P sub 1" is how P_1 is usually read.) Then it is clear that right triangles OPQ and OP_1Q_1 are similar, and hence pairs of corresponding ratios, such as $\frac{|PQ|}{|OP|}$ and $\frac{|P_1Q_1|}{|OP_1|}$, are all equal. Therefore, all the trigonometric functions are indeed well defined.

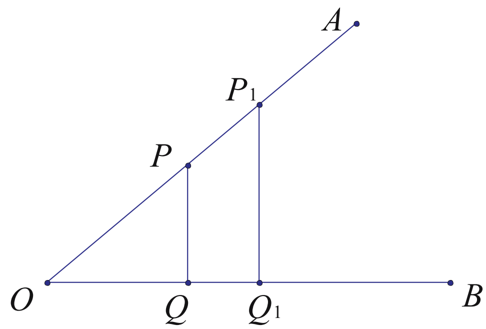

Figure 1.1.

By the above definitions, it is not difficult to see that $\sin\theta$, $\cos\theta$, and $\tan\theta$ are the reciprocals of $\csc\theta$, $\sec\theta$, and $\cot\theta$, respectively. Hence for most purposes, it is enough to consider $\sin\theta$, $\cos\theta$, and $\tan\theta$. It is also not difficult to see that

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \quad \text{and} \quad \frac{\cos\theta}{\sin\theta} = \cot\theta.$$

By convention, in triangle ABC, we let a, b, c denote the lengths of sides BC, CA, and AB, and let $\angle A$, $\angle B$, and $\angle C$ denote the angles CAB, ABC, and BCA. Now, consider a right triangle ABC with $\angle C = 90^\circ$ (Figure 1.2).

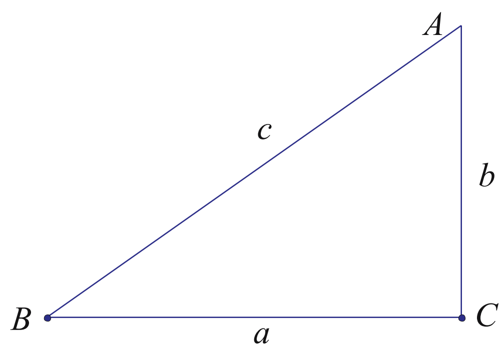

Figure 1.2.

For abbreviation, we write $\sin A$ for $\sin\angle A$. We have

$$\sin A = \frac{a}{c}, \qquad \cos A = \frac{b}{c}, \qquad \tan A = \frac{a}{b};$$
$$\sin B = \frac{b}{c}, \qquad \cos B = \frac{a}{c}, \qquad \tan B = \frac{b}{a};$$

and

$$\begin{aligned} &a = c\sin A, \quad && a = c\cos B, \quad && a = b\tan A; \\ &b = c\sin B, && b = c\cos A, && b = a\tan B; \\ &c = a\csc A, && c = a\sec B, && c = b\csc B, \quad c = b\sec A. \end{aligned}$$

It is then not difficult to see that if A and B are two angles with $0^\circ < A, B < 90^\circ$ and $A + B = 90^\circ$, then $\sin A = \cos B$, $\cos A = \sin B$, $\tan A = \cot B$, and $\cot A = \tan B$. In the right triangle ABC, we have $a^2 + b^2 = c^2$. It follows that

$$(\sin A)^2 + (\cos A)^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1.$$

It can be confusing to write $(\sin A)^2$ as $\sin A^2$. (Why?) For abbreviation, we write $(\sin A)^2$ as $\sin^2 A$. We have shown that for $0^\circ < A < 90^\circ$,

$$\sin^2 A + \cos^2 A = 1.$$

Dividing both sides of the above equation by $\sin^2 A$ gives

$$1 + \cot^2 A = \csc^2 A, \quad \text{or} \quad \csc^2 A - \cot^2 A = 1.$$

Similarly, we can obtain

$$\tan^2 A + 1 = \sec^2 A, \quad \text{or} \quad \sec^2 A - \tan^2 A = 1.$$

Now we consider a few special angles.

In triangle ABC, suppose $\angle A = \angle B = 45^\circ$, and hence $|AC| = |BC|$ (Figure 1.3, left). Then $c^2 = a^2 + b^2 = 2a^2$, and so $\sin 45^\circ = \sin A = \frac{a}{c} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Likewise, we have $\cos 45^\circ = \frac{\sqrt{2}}{2}$ and $\tan 45^\circ = \cot 45^\circ = 1$.

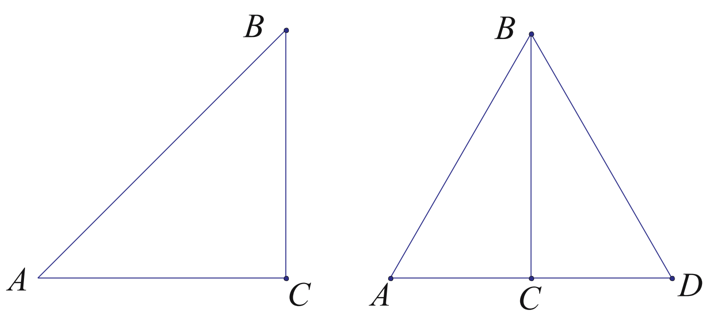

Figure 1.3.

In triangle ABC, suppose $\angle A = 60^\circ$ and $\angle B = 30^\circ$ (Figure 1.3, right). We reflect A across line BC to point D. By symmetry, $\angle D = 60^\circ$, so triangle ABD is equilateral. Hence, $|AD| = |AB|$ and $|AC| = \frac{|AD|}{2}$. Because ABC is a right triangle, $|AB|^2 = |AC|^2 + |BC|^2$. So we have $|BC|^2 = |AB|^2 - \frac{|AB|^2}{4} = \frac{3|AB|^2}{4}$, or $|BC| = \frac{\sqrt{3}|AB|}{2}$. It follows that $\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$, $\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$, $\tan 30^\circ = \cot 60^\circ = \frac{\sqrt{3}}{3}$, and $\tan 60^\circ = \cot 30^\circ = \sqrt{3}$.

We provide one exercise for the reader to practice with right-triangle trigonometric functions. In triangle ABC (see Figure 1.4), $\angle BCA = 90^\circ$, and D is the foot of the perpendicular line segment from C to segment AB. Given that $|AB| = x$ and $\angle A = \theta$, express all the lengths of the segments in Figure 1.4 in terms of x and θ.

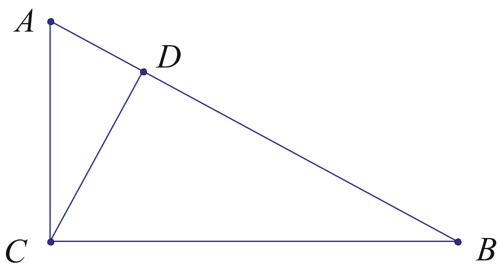

Figure 1.4.

Think Within the Box

For two angles α (Greek "alpha") and β (Greek "beta") with $0^\circ < \alpha, \beta, \alpha + \beta < 90^\circ$, it is not difficult to note that the trigonometric functions do not satisfy the additive distributive law; that is, identities such as $\sin(\alpha + \beta) = \sin\alpha + \sin\beta$ and $\cos(\alpha+\beta) = \cos\alpha + \cos\beta$ are not true. For example, setting $\alpha = \beta = 30^\circ$, we have $\cos(\alpha + \beta) = \cos 60^\circ = \frac{1}{2}$, which is not equal to $\cos\alpha + \cos\beta = 2\cos 30^\circ = \sqrt{3}$. Naturally, we might ask ourselves questions such as how $\sin\alpha$, $\sin\beta$, and $\sin(\alpha+\beta)$ relate to one another.

Consider the diagram of Figure 1.5. Let DEF be a right triangle with $\angle DEF = 90^\circ$, $\angle FDE = \beta$, and $|DF| = 1$ inscribed in the rectangle $ABCD$. (This can always be done in the following way. Construct line ℓ_1 passing through D outside of triangle DEF such that lines ℓ_1 and DE form an acute angle congruent to α. Construct line ℓ_2 passing through D and perpendicular to line ℓ_1. Then A is the foot of the perpendicular from E to line ℓ_1, and C the foot of the perpendicular from F to ℓ_2. Point B is the intersection of lines AE and CF.)

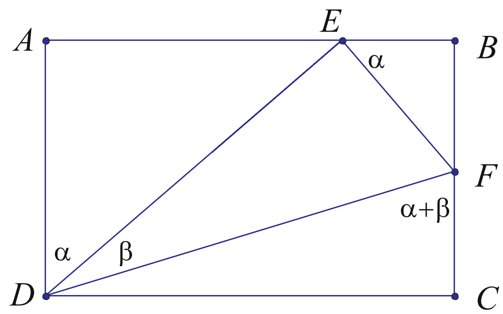

Figure 1.5.

We compute the lengths of the segments inside this rectangle. In triangle DEF, we have $|DE| = |DF| \cdot \cos\beta = \cos\beta$ and $|EF| = |DF| \cdot \sin\beta = \sin\beta$. In triangle ADE, $|AD| = |DE| \cdot \cos\alpha = \cos\alpha\cos\beta$ and $|AE| = |DE| \cdot \sin\alpha = \sin\alpha\cos\beta$. Because $\angle DEF = 90^\circ$, it follows that $\angle AED + \angle BEF = 90^\circ = \angle AED + \angle ADE$, and so $\angle BEF = \angle ADE = \alpha$. (Alternatively, one may observe that right triangles ADE and BEF are similar to each other.) In triangle BEF, we have $|BE| = |EF| \cdot \cos\alpha = \cos\alpha\sin\beta$ and $|BF| = |EF| \cdot \sin\alpha = \sin\alpha\sin\beta$. Since $AD \parallel BC$, $\angle DFC = \angle ADF = \alpha + \beta$. In right triangle CDF, $|CD| = |DF| \cdot \sin(\alpha+\beta) = \sin(\alpha+\beta)$ and $|CF| = |DF| \cdot \cos(\alpha+\beta) = \cos(\alpha+\beta)$.

From the above, we conclude that

$$\cos\alpha\cos\beta = |AD| = |BC| = |BF| + |FC| = \sin\alpha\sin\beta + \cos(\alpha+\beta),$$

implying that

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Similarly, we have

$$\sin(\alpha+\beta) = |CD| = |AB| = |AE| + |EB| = \sin\alpha\cos\beta + \cos\alpha\sin\beta;$$

that is,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

By the definition of the tangent function, we obtain

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} \\ &= \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}.\end{aligned}$$

We have thus proven the **addition formulas** for the sine, cosine, and tangent functions for angles in a restricted interval. In a similar way, we can develop an addition formula for the cotangent function. We leave it as an exercise.