83. Solving an Equation for an Unknown Function In Exercise 65 of Section 2.7 you were asked to solve equations in which the unknowns were functions. Now that we know about inverses and the identity function (see Exercise 82), we can use algebra to solve such equations. For instance, to solve $f \circ g = h$ for the unknown function f, we perform the following steps:

$f \circ g = h$	Problem: Solve for f
$f \circ g \circ g^{-1} = h \circ g^{-1}$	Compose with g^{-1} on the right
$f \circ I = h \circ g^{-1}$	$g \circ g^{-1} = I$
$f = h \circ g^{-1}$	$f \circ I = f$

So the solution is $f = h \circ g^{-1}$. Use this technique to solve the equation $f \circ g = h$ for the indicated unknown function.

(a) Solve for f, where $g(x) = 2x + 1$ and $h(x) = 4x^2 + 4x + 7$

(b) Solve for g, where $f(x) = 3x + 5$ and $h(x) = 3x^2 + 3x + 2$

$$f^{-1} \circ f = x$$

$$= \quad f(g(x)) = h(x)$$

$$f(g(g^{-1}(x))) = h(g^{-1}(x))$$

$$f(x) = h(g^{-1}(x))$$

a) $f(g(x)) = h(x)$ / $g^{-1}(x) = \frac{x-1}{2}$

$$f(2x+1) = 4x^2+4x+7$$

$$f\left(2\left(\frac{x-1}{2}\right)+1\right) = 4\left(\frac{x-1}{2}\right)^2 + 4\left(\frac{x-1}{2}\right) + 7$$

$$f(x) = (x-1)^2 + 2(x-1) + 7$$

$$= x^2 - 2x + 1 + 2x - 2 + 7 = x^2 + 6$$

---

a) $h(x) = \underbrace{4x^2 + 4x + 1} + 6$

$$= (2x+1)^2 + 6$$

$$g(x) = 2x+1$$

$$= (g(x))^2 + 6 \rightarrow f(x) = x^2 + 6$$

$f(x) = 3x + 5 \rightarrow f^{-1}(x) = \frac{x-5}{3}$

$h(x) = 3x^2 + 3x + 2$

$f^{-1}(f(g(x))) = f^{-1}(h(x))$

$g(x) = f^{-1}(h(x))$

$= f^{-1}(3x^2+3x+2)$

$= \frac{(3x^2+3x+2)-5}{3}$

$= x^2 + x - 1$

$g(x) = f^{-1}(h(x))$

$h(x) = 3x^2 + 3x + 2 + 3 - 3$

$= 3x^2 + 3x - 3 + 5$

$= 3(x^2 + x - 1) + 5$

$f(g(x)) = h(x)$

$3(g(x)) + 5 = 3x^2 + 3x + 2$

$3\,g(x) = 3x^2 + 3x - 3$

$g(x) = x^2 + x - 1$

$h(x) = f(g(x))$

$h(x) = \sqrt{3x-1} + 5$

1) if $f(x) = \sqrt{x} + 5$, find $g(x)$.

$g(x) = 3x-1$

2) If $g(x) = 3x$, Find $f(x)$.

$f(x) = \sqrt{x-1} + 5$

$h(x) = f(g(x))$

$h(x) = \sqrt{x-3}$

$g(x) = e^{x-5} + 4$

Find $f(x) = h(g^{-1}(x))$

$y = e^{x-5} + 4 \xrightarrow{\text{inv}} x = e^{y-5} + 4$

$x - 4 = e^{y-5}$

$\ln|x-4| = y - 5$

$g^{-1}(x) = \ln|x-4| + 5 = y$

$f(x) = h(\ln|x-4| + 5)$

$= \sqrt{(\ln|x-4| + 5) - 3}$

$= \sqrt{\ln|x-4| + 2}$