12 Positive real numbers $x \neq 1$ and $y \neq 1$ satisfy $\log_2 x = \log_y 16$ and $xy = 64$. What is $(\log_2 \frac{x}{y})^2$?

(A) $\frac{25}{2}$ **(B)** 20 **(C)** $\frac{45}{2}$ **(D)** 25 **(E)** 32

$\log_2 x = 4\log_y 2$

$xy=64$

$y=\frac{64}{x}$

$\left(\log_2 x-\log_2 y\right)^2$

$\log_2 x = 4\log_{\frac{64}{x}} 2$

$\frac{\log x}{\log 2} = 4\,\frac{\log 2}{\log 64 - \log x}$ $\rightarrow$ $4(\log 2)^2 = 6\log 2\log x - (\log x)^2$

$\log 64 \downarrow 6\log 2$

12 Positive real numbers $x \neq 1$ and $y \neq 1$ satisfy $\log_2 x = \log_y 16$ and $xy = 64$. What is $(\log_2 \frac{x}{y})^2$?

(A) $\frac{25}{2}$ **(B)** 20 **(C)** $\frac{45}{2}$ **(D)** 25 **(E)** 32

$xy = 2^k 2^{\frac{4}{k}} = 64 = 2^6$

$k = \log_2 x = \log_y 16$

$k + \frac{4}{k} = 6$

$2^k = x \quad y^k = 16$

$\cancel{k^2 - 6k + 4 = 0}$

$y = 16^{\frac{1}{k}} = 2^{\frac{4}{k}}$

$\cancel{k = 3 \pm \sqrt{5}}$

$\frac{x}{y} = \frac{2^k}{2^{\frac{4}{k}}} = 2^{k - \frac{4}{k}}$

$\log_2 \frac{x}{y}$

$= \log_2 2^{k - \frac{4}{k}}$

$\left(k + \frac{4}{k}\right)^2 = (6)^2$

$\left(k - \frac{4}{k}\right)^2$

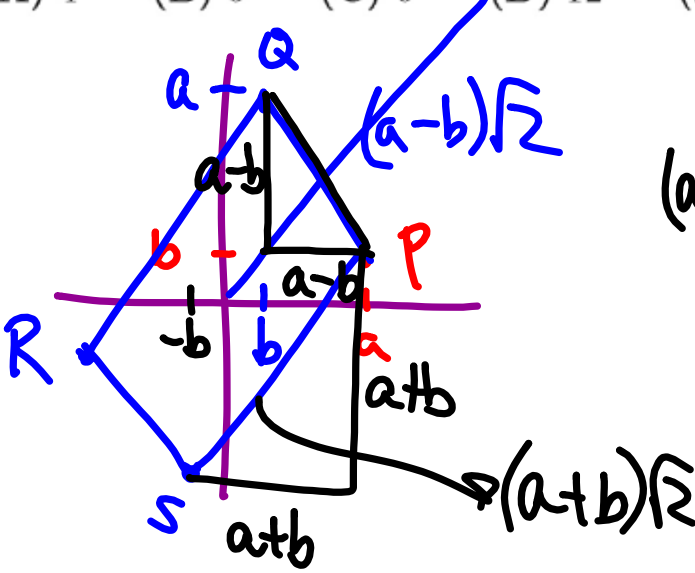
$k - \frac{4}{k}$

$k^2 + 8 + \frac{16}{k^2} = 36$

$= k^2 - 8 + \frac{16}{k^2} \rightarrow 20$

10. A quadrilateral has vertices $P(a, b)$, $Q(b, a)$, $R(-a, -b)$, and $S(-b, -a)$, where a and b are integers with $a > b > 0$. The area of $PQRS$ is 16. What is $a + b$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 12 **(E)** 13



$(a-b)\sqrt{2}(a+b)\sqrt{2} = 16$

$a^2 - b^2 = 8$

$3^2 - 1^2$

$a+b=4$

9 A sequence of numbers is defined recursively by $a_1 = 1$, $a_2 = \frac{3}{7}$, and

$$a_n = \frac{a_{n-2} \cdot a_{n-1}}{2a_{n-2} - a_{n-1}}$$

for all $n \geq 3$ Then a_{2019} can be written as $\frac{p}{q}$, where p and q are relatively prime positive inegers. What is $p + q$?

(A) 2020 **(B)** 4039 **(C)** 6057 **(D)** 6061 **(E)** 8078

$$a_3 = \frac{1 \cdot \frac{3}{7}}{2 \cdot 1 - \frac{3}{7}} = \frac{3}{11}$$

$$a_4 = \frac{\frac{3}{7} \cdot \frac{3}{11}}{2 \cdot \frac{3}{7} - \frac{3}{11}} = \frac{9}{45} = \frac{3}{15}$$

$4n - 1$

$4(2019) - 1$

$$\frac{3}{8075}$$

Problem 10

There is a unique positive integer n such that

$$\log_2(\log_{16} n) = \log_4(\log_4 n).$$

What is the sum of the digits of n?

$\textbf{(A)}\ 4 \qquad \textbf{(B)}\ 7 \qquad \textbf{(C)}\ 8 \qquad \textbf{(D)}\ 11 \qquad \textbf{(E)}\ 13$

$= \frac{1}{2}\log_2 \log_4 n$

$\log_{16} n = (\log_4 n)^{1/2}$

$\frac{1}{2}\underbrace{\log_4 n}_{} = (\log_4 n)^{1/2}$

$\frac{1}{4}k^2 = k$

$k^2 - 4k = 0$

$k = 0, 4$

$\log_4 n = 4$

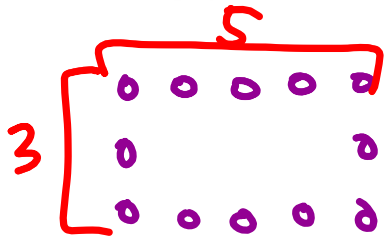
$n = 256$

$\boxed{13}$

February 10, 2020

9. Carl decided to fence in his rectangular garden. He bought 20 fence posts, placed one on each of the four corners, and spaced out the rest evenly along the edges of the garden, leaving exactly 4 yards between neighboring posts. The longer side of his garden, including the corners, has twice as many posts as the shorter side, including the corners. What is the area, in square yards, of Carl's garden?

(A) 256 **(B)** 336 **(C)** 384 **(D)** 448 **(E)** 512



$4 \overline{)28}$ = 8

12

12