November 1, 2019

$\boxed{1}$ **DEFINITION** Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} \stackrel{?}{=} 0$$

$$\sqrt{x^2+y^2} < \delta \qquad \left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon$$

$$\left|\frac{3x^2y}{x^2+y^2}\right| = \frac{3x^2|y|}{x^2+y^2} \le 3|y| = 3\sqrt{y^2} \le 3\sqrt{x^2+y^2}$$

$$\frac{x^2}{x^2+y^2} \to \le 1$$

Let $\delta = \frac{\varepsilon}{3}$

We know $\sqrt{x^2+y^2} < \delta$

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| = \frac{3x^2|y|}{x^2+y^2} \le 3|y| \le 3\sqrt{x^2+y^2} < 3\delta = \varepsilon$$

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \varepsilon$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$$

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} \stackrel{?}{=} 0$$

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|x||y|}{\sqrt{x^2+y^2}}$$

$$= \sqrt{\frac{x^2}{x^2+y^2}}\,|y|$$

$$\sqrt{x^2+y^2} < \delta$$

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon$$

$$\frac{|x||y|}{\sqrt{x^2+y^2}} = \frac{\sqrt{x^2}}{\sqrt{x^2+y^2}}|y| \le |y| = \sqrt{y^2} \le \sqrt{x^2+y^2} < \varepsilon$$

$$\le 10\sqrt{x^2+y^2}$$

$$\sqrt{x^2+y^2} < \delta \qquad \delta = \varepsilon$$

$$\frac{|xy|}{\sqrt{x^2+y^2}} \le |y| = \sqrt{y^2} \le \sqrt{x^2+y^2} < \varepsilon$$

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

Q.E.D.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin y}{x^2 + 2y^2}$$

$2x + 4 = 10$

$2x = 6$

$x = 3$

$40x + 80 = 200$