

I DEFINITION Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b) . Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if } (x, y) \in D \text{ and } 0 < \sqrt{(x - a)^2 + (y - b)^2} < \delta \text{ then } |f(x, y) - L| < \varepsilon$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2 + y^2} \stackrel{?}{=} 0$$

$$\sqrt{x^2 + y^2} < \delta$$

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \varepsilon$$

$$\left| \frac{3x^2y}{x^2 + y^2} \right| = \frac{3x^2 |y|}{x^2 + y^2} \leq 3 |y| = 3 \sqrt{y^2} \leq 3 \sqrt{x^2 + y^2}$$

≤ 1

$$\text{Let } \delta = \frac{\varepsilon}{3}$$

$$\text{We know } \sqrt{x^2 + y^2} < \delta$$

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| = \frac{3x^2 |y|}{x^2 + y^2} \leq 3 |y| \leq 3 \sqrt{x^2 + y^2} < 3\delta = \varepsilon$$

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \varepsilon$$

$$\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2 + y^2} = 0$$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2 + y^2}} \stackrel{?}{=} 0$$

$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \frac{|x||y|}{\sqrt{x^2+y^2}}$
 $= \sqrt{\frac{x^2}{x^2+y^2}} |y|$
 $\frac{|x||y|}{\sqrt{x^2+y^2}} = \frac{|x|}{\sqrt{x^2+y^2}} |y| \leq |y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} < \epsilon$
 $\sqrt{x^2+y^2} < \delta \quad \delta = \epsilon$
 $\frac{|xy|}{\sqrt{x^2+y^2}} \leq |y| \sqrt{x^2+y^2} \leq \sqrt{x^2+y^2} < \epsilon$
 $\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| < \epsilon$
 $\therefore \lim_{(x,y) \rightarrow (0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$
\$\hookrightarrow \leq 10 \sqrt{x^2+y^2}\$
\$\downarrow\$
Q.E.D.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 \sin y}{x^2 + 2y^2}$$

$$\begin{aligned}2x + 4 &= 10 \\2x &= 6 \\x &= 3\end{aligned}$$
$$40x + 30 = 200$$

The diagram shows handwritten red annotations on a white background. At the top left, there is a curved red arrow pointing from the first equation $2x + 4 = 10$ towards the second equation $40x + 30 = 200$. Below the first equation, there is a vertical red arrow pointing downwards to the solution $x = 3$.