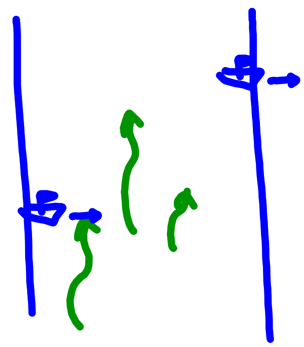
31. Water traveling along a straight portion of a river normally flows fastest in the middle, and the speed slows to almost zero at the banks. Consider a long straight stretch of river flowing north, with parallel banks 40 m apart. If the maximum water speed is 3 m/s, we can use a quadratic function as a basic model for the rate of water flow x units from the west bank: $f(x) = \frac{3}{400}x(40 - x)$.

(a) A boat proceeds at a constant speed of 5 m/s from a point A on the west bank while maintaining a heading perpendicular to the bank. How far down the river on the opposite bank will the boat touch shore? Graph the path of the boat.



$$V = \left\langle 5, \frac{3}{400}x(40-x) \right\rangle \qquad \int \frac{3}{400}x(40-x)\,dt$$

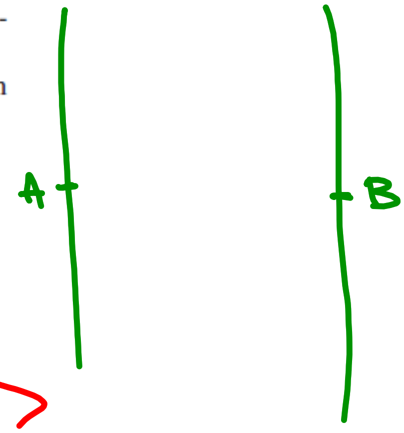
$$r = \left\langle 5t, \frac{3t^2}{4} - \frac{t^3}{16} \right\rangle \qquad = \int \frac{3}{400}(5t)(40-5t)\,dt$$

$$5t = 40$$

$$t = 8, \quad \left.\frac{3t^2}{4} - \frac{t^3}{16}\right|_{t=8} \qquad = \int \frac{3}{2}t - \frac{3}{16}t^2\,dt$$

$$= 48 - 32 = \underline{\underline{16}}$$

(b) Suppose we would like to pilot the boat to land at the point B on the east bank directly opposite A. If we maintain a constant speed of 5 m/s and a constant heading, find the angle at which the boat should head. Then graph the actual path the boat follows. Does the path seem realistic?



$V_x = 5\cos\theta$

$V_y = -5\sin\theta$

$$V = \langle 5\cos\theta, -5\sin\theta + \frac{3}{400}x(40-x) \rangle$$

$$r = \langle 5\cos\theta t, -5\sin\theta t + \int \frac{3}{400}(5\cos\theta t)(40 - 5\cos\theta t)\,dt \rangle$$

$\cos\theta = \frac{8}{t}$

$\sin\theta = \frac{\sqrt{t^2 - 8^2}}{t}$

$$-5\sin\theta t + \int \frac{3}{2}\cos\theta t - \frac{3}{16}t^2\cos^2\theta\, dt$$

$$-5\sin\theta t + \cos\theta\left(\frac{3t^2}{4} - \frac{t^3}{16}\cos\theta\right) = 0$$

$$2t$$

$$-5\sqrt{t^2-64} + \frac{\cancel{8}^{\,2}}{\cancel{t}}\left(\frac{3t^2}{4} - \frac{\cancel{t^3}^{\,t^2}}{\cancel{16}}\cdot\frac{\cancel{8}}{\cancel{t}}\right) = 0$$

$$\frac{\cancel{t^2}}{\cancel{4}} \quad 2$$

$2t = 5\sqrt{t^2-64}$

$\frac{4t^2}{25} = t^2 - 64$

$25 \cdot 64 = 21t^2$

$\frac{40}{\sqrt{21}} = t$

$\cos\theta = \frac{8}{\frac{40}{\sqrt{21}}} = \frac{\sqrt{21}}{5}$

$\theta = 23.6^\circ$