10 Inequalities

Just as we had equations and systems of equations, we can have inequalities and systems of inequalities.

The only difference is that you must reverse the sign every time you either multiply or divide both sides by a negative number.

For example,

$$2x + 3 < 9$$

Do we have to reverse the sign at any point? Well, we would subtract by 3 to get $2x < 6$ and then divide by 2 to get $x < 3$. Yes, we did a subtraction but at no point did we multiply or divide by a negative number. Therefore, the sign stays the same.

Let's take another example:

$$3x + 5 < 4x + 4$$

The first step is to combine like terms. We subtract both sides by $4x$ to get the x's on the left hand side. We then subtract both sides by 5 to get the constants on the right hand side:

$$3x - 4x < 4 - 5$$
$$-x < -1$$

Notice that the sign hasn't changed yet. Now, to get rid of the negative in front of the x, we need to **multiply** both sides by -1. Doing so means we need to reverse the sign.

$$x > 1$$

This concept is the cause of so many silly mistakes that it's important to reiterate it. Just working with negative numbers does NOT mean you need to change the sign. Some students see that they're dividing a negative number and impulsively reverse the sign. Don't do that. Only reverse the sign when you multiply or divide both sides **by** a negative number.

EXAMPLE 1: Which of the following integers is a solution to the inequality $-3x - 7 \leq -7x - 27$?

A) -6 B) -3 C) 1 D) 4

$$\begin{aligned} -3x - 7 &\leq -7x - 27 \\ 4x &\leq -20 \\ x &\leq -5 \end{aligned}$$

At no point did we multiply or divide by a negative number so there was no need to reverse the sign. We divided a negative number, -20, but we did so by a positive number, 4.

The only answer choice that satisfies $x \leq -5$ is -6, answer $\boxed{(A)}$.

EXAMPLE 2: If $-7 \leq -2x + 3 \leq 15$, which of the following must be true?

A) $5 \leq x \leq 6$ B) $-6 \leq x \leq -5$ C) $-6 \leq x \leq 5$ D) $-5 \leq x \leq 6$

So how do we solve these "two-inequalities-in-one" problems? Well, we can split them up into two inequalities that we can solve separately:

$$-7 \leq -2x + 3$$

$$-2x + 3 \leq 15$$

Solving the first inequality,

$$\begin{aligned} -7 &\leq -2x + 3 \\ -10 &\leq -2x \\ 5 &\geq x \end{aligned}$$

Solving the second inequality,

$$\begin{aligned} -2x + 3 &\leq 15 \\ -2x &\leq 12 \\ x &\geq -6 \end{aligned}$$

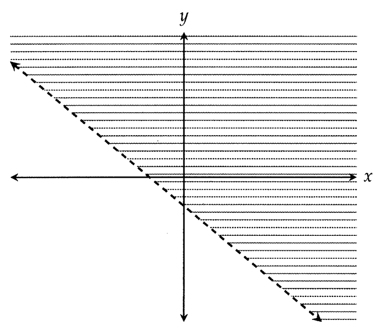
Putting the two results together, we get $-6 \leq x \leq 5$. Answer $\boxed{(C)}$.

EXAMPLE 3: To follow his diet plan, James must limit his daily sugar consumption to at most 40 grams. One cookie has 5 grams of sugar and one fruit salad contains 7 grams of sugar. If James ate only cookies and fruit salads, which of the following inequalities represents the possible number of cookies c and fruit salads s that he could eat in one day and remain within his diet's sugar limit?

A) $\frac{5}{c} + \frac{7}{s} < 40$ B) $\frac{5}{c} + \frac{7}{s} \leq 40$ C) $5c + 7s < 40$ D) $5c + 7s \leq 40$

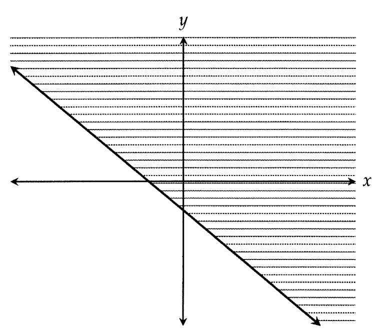
The total amount of sugar he gets from cookies is $5c$. The total amount of sugar he gets from fruit salads is $7s$. So his total sugar intake for any given day is $5c + 7s$, and since it can't be more than 40 grams, $5c + 7s \leq 40$. Answer $\boxed{(D)}$.

From a graphing standpoint, what does an inequality look like? What does it mean for $y > -x - 1$?



As shown by the shaded region above, the inequality $y > -x - 1$ represents all the points above the line $y = -x - 1$. If you have a hard time keeping track of what's above a line and what's below, just look at the y-axis. The line cuts the y-axis into two parts. The top part of the y-axis is always in the "above" region. The bottom part of the y-axis is always in the "below" region. If the graph doesn't show the intersection with the y-axis, you can always just draw your own vertical line through the graph to determine the "above" and "below" regions.

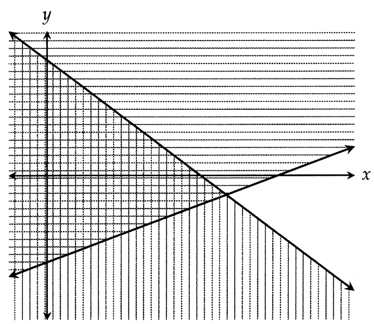
Also note that the line is dashed. Because $y > -x - 1$ and NOT $y = -x - 1$, the points on the line itself do not satisfy the inequality. If the equation were $y \geq -x - 1$, then the line would be solid, and points on the line would satisfy the inequality.



But what about a system of inequalities?

$$y \leq -x + 4$$
$$y \geq \frac{1}{2}x - 3$$

When it comes to graphing, the goal is to find the region with the points that satisfy the system. In this case, we want all the points below $y = -x + 4$ but above $y = \frac{1}{2}x - 3$. We can shade the regions below $y = -x + 4$ and above $y = \frac{1}{2}x - 3$ to see where the shaded regions overlap. The overlapping region will contain all the points that satisfy both inequalities.



$$y \leq -x + 4$$
$$y \geq \frac{1}{2}x - 3$$

The overlapping region on the left represents all the solutions to the system.

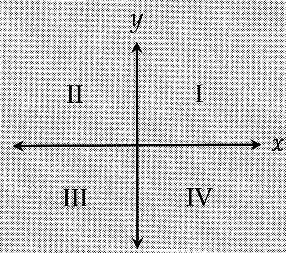
Setting the two equations equal to each other and solving gives the intersection point, which, in this case, happens to be the solution with the highest value of x.

$$-x + 4 = \frac{1}{2}x - 3$$
$$-2x + 8 = x - 6$$
$$-3x = -14$$
$$x = \frac{-14}{-3} \approx 4.66$$

At $x = 4.66$, $y = -4.66 + 4 = -0.66$ (we get this from the first equation). Therefore, $(4.66, -0.66)$ is the solution with the highest value of x. There are no solutions in which x is 5, 6, or larger.

While finding the intersection point in this example may have seemed a bit pointless (haha!), these points can be very important in the context of a given situation, such as finding the right price to maximize profit or figuring out the right materials for a construction project.

EXAMPLE 4:



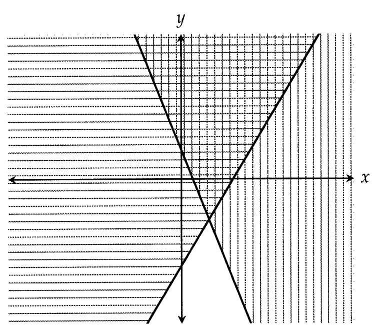
The following system of inequalities is graphed in the xy-plane above.

$$y \geq -3x + 1$$
$$y \geq 2x - 3$$

Which quadrants contain solutions to the system?

A) Quadrants I and II B) Quadrants I and IV C) Quadrants III and IV D) Quadrants I, II, and IV

First, graph the equations, preferably with your graphing calculator. Then shade the regions and find the overlapping region.



As you can see, the overlapping region, which contains all the solutions, is the top region. It has points in quadrants I, II, and IV. Answer $\boxed{(D)}$.

EXAMPLE 5: Ecologists have determined that the number of frogs y must be greater than or equal to three times the number of snakes x for a healthy ecosystem to be maintained in a particular forest. In addition, the number of frogs and the number of snakes must sum to at least 400.

PART 1: Which of the following systems of inequalities expresses these conditions for a healthy ecosystem?

A) $y \geq 3x$, $y - x > 400$

B) $y \geq 3x$, $y - x \geq 400$

C) $y \geq 3x$, $y + x \geq 400$

D) $y \leq 3x$, $y + x \leq 400$

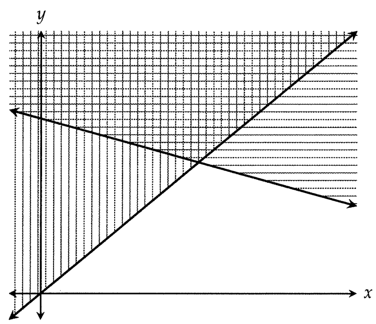
PART 2: What is the minimum possible number of frogs in a healthy ecosystem?

Part 1 Solution: The number of frogs, y, must be at least three times the number of snakes, x. So, $y \geq 3x$. The number of frogs and the number of snakes must sum to at least 400, so $y + x \geq 400$. Answer $\boxed{(C)}$.

Part 2 Solution: In these types of questions, the intersection point is typically what we're looking for, but we'll graph the inequalities just to make sure. First, put the second inequality into $y = mx + b$ form.

$$y \geq 3x$$
$$y \geq -x + 400$$

Then graph the inequalities using your calculator.



The graph confirms that y, the number of frogs, is at a minimum at the intersection point. After all, the overlapping region (the top region) represents all possible solutions and the intersection point is at the bottom of this region, representing the solution with the minimum number of frogs.

We can find the coordinates of that intersection point by solving a system of equations based on the two lines.

$$y = 3x$$
$$y = -x + 400$$

Substituting the first equation into the second,

$$3x = -x + 400$$
$$4x = 400$$
$$x = 100$$

100 is the x-coordinate. The y-coordinate is $y = 3x = 3(100) = 300$. The intersection point is at $(100, 300)$ and the minimum number of frogs is $\boxed{300}$ in a healthy ecosystem.

CHAPTER EXERCISE: Answers for this chapter start on page 276.

A calculator is allowed on the following questions.

1

Which of the following is a solution to the inequality $-x - 4 > 4x - 14$?

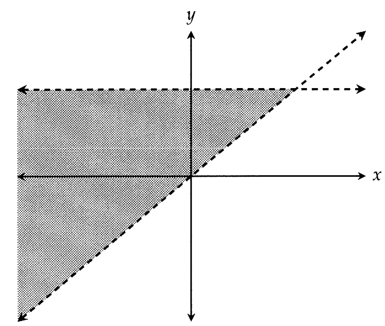
A) -1
B) 2
C) 5
D) 8

2

If $\frac{3}{4}x - 4 > \frac{1}{2}x - 10$, which of the following must be true?

A) $x < 24$
B) $x > 24$
C) $x < -24$
D) $x > -24$

3



Which of the following systems of inequalities could be the one graphed in the xy-plane above?

A) $y > 3$
$y > x$

B) $y < 3$
$y < x$

C) $y < 3$
$y > x$

D) $y > 3$
$y < x$

4

Jerry estimates that there are m marbles in a jar. Harry, who knows the actual number of marbles in the jar, notes that the actual number, n, is within 10 marbles (inclusive) of Jerry's estimate. Which of the following inequalities represents the relationship between Jerry's estimate and the actual number of marbles in the jar?

A) $n + 10 \leq m \leq n - 10$
B) $m - 10 \leq n \leq m + 10$
C) $n \leq m \leq 10n$
D) $\frac{m}{10} \leq n \leq 10m$

5

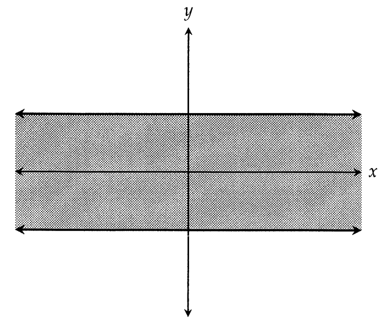
A manufacturer produces chairs for a retail store according to the formula, $M = 12P + 100$, where M is the number of units produced and P is the retail price of each chair. The number of units sold by the retail store is given by $N = -3P + 970$, where N is the number of units sold and P is the retail price of each chair. What are all the values of P for which the number of units produced is greater than or equal to the number of units sold?

A) $P \geq 58$
B) $P \leq 58$
C) $P \geq 55$
D) $P \leq 55$

6

If n is an integer and $3(n-2) > -4(n-9)$, what is the least possible value of n?

7



The graph in the xy-plane above could represent which of the following systems of inequalities?

A) $y \geq 3$
$y \leq -3$

B) $y \leq 3$
$y \geq -3$

C) $x \geq 3$
$x \leq -3$

D) $x \leq 3$
$x \geq -3$

8

To get to work, Harry must travel 8 miles by bus and 16 miles by train everyday. The bus travels at an average speed of x miles per hour and the train travels at an average speed of y miles per hour. If Harry's daily commute never takes more than 1 hour, which of the following inequalities represents the possible average speeds of the bus and train during the commute?

A) $\frac{8}{x} + \frac{16}{y} \leq 1$
B) $\frac{16}{x} + \frac{8}{y} \leq 1$
C) $\frac{x}{8} + \frac{y}{16} \leq 1$
D) $8x + 16y \leq 1$

9

An ice cream distributor contracts out to two different companies to manufacture cartons of ice cream. Company A can produce 80 cartons each hour and Company B can produce 140 cartons each hour. The distributor needs to fulfill an order of over 1,100 cartons in 10 hours of contract time. It contracts out x hours to Company A and the remaining hours to Company B. Which of the following inequalities gives all possible values of x in the context of this problem?

A) $\frac{80}{x} + \frac{140}{10-x} > 1,100$

B) $140x + 80(10 - x) > 1,100$

C) $80x + 140(10 - x) > 1,100$

D) $80x + 140(x - 10) > 1,100$

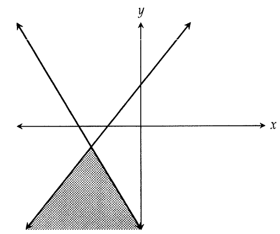
10

$$y \geq \frac{3}{2}x + 2$$

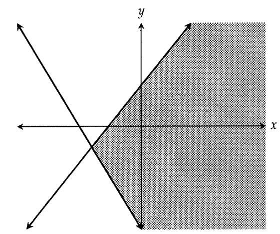
$$y \leq -2x - 5$$

Which of the following graphs in the xy-plane could represent the system of inequalities above?

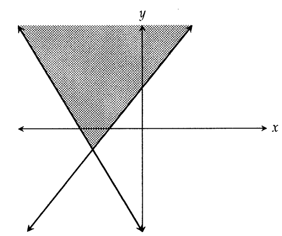
A)



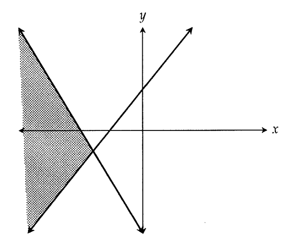
B)



C)



D)



11

$$y > 15x + a$$
$$y < 5x + b$$

In the system of inequalities above, a and b are constants. If $(1, 20)$ is a solution to the system, which of the following could be the value of $b - a$?

A) 6

B) 8

C) 10

D) 12

12

Tina works no more than 30 hours at a nail salon each week. She can do a manicure in 20 minutes and a pedicure in 30 minutes. Each manicure earns her \$25 and each pedicure earns her \$40, and she must earn at least \$900 to cover her expenses. If during one week, she does enough manicures m and pedicures p to cover her expenses, which of the following systems of inequalities describes her working hours and her earnings?

A) $3m + 2p \leq 30$
$25m + 40p \geq 900$

B) $2m + 3p \leq 30$
$25m + 40p \geq 900$

C) $\frac{m}{3} + \frac{p}{2} \leq 30$
$25m + 40p \geq 900$

D) $\frac{m}{3} + \frac{p}{2} \geq 900$
$25m + 40p \leq 30$

13

If $k \leq x \leq 3k + 12$, which of the following must be true?

I. $x - 12 \leq 3k$
II. $k \geq -6$
III. $x - k \geq 0$

A) I only

B) I and II only

C) II and III only

D) I, II, and III

14

If $-\frac{20}{3} < -2x + 4 < -\frac{9}{2}$, what is one possible value of $x - 2$?

15

Joyce wants to create a rectangular garden that has an area of at least 300 square meters and a perimeter of at least 70 meters. If the length of the garden is x meters long and the width is y meters long, which of the following systems of inequalities represents Joyce's requirements?

A) $xy \geq 70$
$x + y \geq 300$

B) $xy \geq 150$
$x + y \geq 70$

C) $xy \geq 300$
$x + y \geq 70$

D) $xy \geq 300$
$x + y \geq 35$

16

If $a < b$, which of the following must be true?

I. $a^2 < b^2$
II. $2a < 2b$
III. $-b < -a$

A) II only
B) I and II only
C) II and III only
D) I, II, and III

Chapter 10: Inequalities

CHAPTER EXERCISE:

1. $\boxed{A}$

$$\begin{aligned} -x - 4 &> 4x - 14 \\ -5x &> -10 \\ x &< 2 \end{aligned}$$

Of the answer choices, only -1 is a solution.

2. $\boxed{D}$ Multiply both sides by 4 to get rid of the fractions.

$$\begin{aligned} \frac{3}{4}x - 4 &> \frac{1}{2}x - 10 \\ 3x - 16 &> 2x - 40 \\ x &> -24 \end{aligned}$$

3. $\boxed{C}$ The shaded region falls below the horizontal line $y = 3$, so $y < 3$. The shaded region also stays above $y = x$, so $y > x$.

4. $\boxed{B}$ Let's say Jerry's estimate, m, is 100 marbles. If the actual number of marbles is within 10 of that estimate, then the actual number must be at least 90 and at most 110. Using variables, $m - 10 \leq n \leq m + 10$.

5. $\boxed{A}$ Setting up the inequality,

$$\begin{aligned} M &\geq N \\ 12P + 100 &\geq -3P + 970 \\ 15P &\geq 870 \\ P &\geq 58 \end{aligned}$$

6. $\boxed{7}$

$$\begin{aligned} 3(n-2) &> -4(n-9) \\ 3n - 6 &> -4n + 36 \\ 7n &> 42 \\ n &> 6 \end{aligned}$$

Since n is an integer, the least possible value of n is 7.

7. $\boxed{B}$ The shaded region is below the horizontal line $y = 3$ but above the horizontal line $y = -3$. Therefore, $y \geq -3$ and $y \leq 3$.

8. $\boxed{A}$ The time Harry spends on the bus is $\frac{8}{x}$ hours and the time he spends on the train is $\frac{16}{y}$ hours. Since the total number of hours is never greater than 1, $\frac{8}{x} + \frac{16}{y} \leq 1$.

9. $\boxed{C}$ If the distributor contracts out to Company A for x hours, then it contracts out to Company B for $10 - x$ hours. Company A then produces $80x$ cartons and Company B produces $140(10 - x)$ cartons. Setting up the inequality,

$$80x + 140(10 - x) > 1,100$$

10. $\boxed{D}$ The line going from the bottom-left to the top-right must be $y = \frac{3}{2}x + 2$ and the line going from the top-left to the bottom-right must be $y = -2x + 5$ (based on the slopes and y-intercepts). Answer (D) correctly shades in the region above $y = \frac{3}{2}x + 2$ and below $y = -2x + 5$.

11. $\boxed{D}$ Plug in $x = 1$, $y = 20$ into the first inequality to get $20 > 15 + a$, $5 > a$. Do the same for the second inequality to get $20 < 5 + b$, $15 < b$. So, a is less than 5 and b is greater than 15. The difference between the two must be more than $15 - 5 = 10$. Among the answer choices, 12 is the only one that is greater than 10.

12. $\boxed{C}$ One manicure takes 1/3 of an hour. One pedicure takes 1/2 an hour. The total number of hours she spends doing manicures and pedicures must be less than or equal to 30, so $\frac{1}{3}m + \frac{1}{2}p \leq 30$. She earns $25m$ for the manicures and $40p$ for the pedicures. Altogether, $25m + 40p \geq 900$.

13. $\boxed{D}$ From the given inequality, $x \leq 3k + 12$. Subtracting 12 from both sides gives $x - 12 \leq 3k$, which confirms that I is always true.

    From the given inequality, $3k + 12 \geq k$, which means $2k \geq -12$, $k \geq -6$, so II must also be true.

    From the given inequality, $k \leq x$. Subtracting k from both sides gives $0 \leq x - k$. Therefore, III must also be true.

14. $\boxed{\frac{9}{4} < x < \frac{10}{3}}$ Let's solve these separately. First,

$$-\frac{20}{3} < -2x + 4$$
$$-20 < -6x + 12$$
$$-32 < -6x$$
$$\frac{16}{3} > x$$

Now for the second part,

$$-2x + 4 < -\frac{9}{2}$$
$$-4x + 8 < -9$$
$$-4x < -17$$
$$x > \frac{17}{4}$$

Putting the two results together, $\frac{17}{4} < x < \frac{16}{3}$. Therefore, $\frac{9}{4} < x - 2 < \frac{10}{3}$.

15. $\boxed{D}$ If the area is at least 300, then $xy \geq 300$. The perimeter of the rectangular garden is $2x + 2y$, so $2x + 2y \geq 70$, which reduces to $x + y \geq 35$.

16. $\boxed{C}$ I is not always true because of negative values. Take $a = -5$ and $b = 2$ for example. $a < b$, but $a^2 > b^2$. II is definitely true. It's the equivalent of multiplying both sides by 2. III is also true. It's the equivalent of multiplying both sides by -1, which necessitates a sign change.