AP Calc Review Materials – Part 1

Derivatives:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(fg) = fg' + gf'$$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - fg'}{g^2}$$

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(Arc\sin x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(Arc\tan x) = \frac{1}{1+x^2}$$

Integrals:

$$\int a\,dx = ax + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C.$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \text{ or } -\ln|\cos x| + C$$

$$\int \cot x\,dx = \ln|\sin x| + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C = -\ln|\csc x + \cot x| + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc^2 x\,dx = -\cot x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

A function $y = f(x)$ is <u>continuous</u> at $x = a$ if

i). $f(a)$ exists

ii). $\lim_{x \to a} f(x)$ exists

iii). $\lim_{x \to a} = f(a)$

Limits of Rational Functions as $x \to \pm\infty$

i). $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} = 0$ if the degree of $f(x) <$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{x^2 - 2x}{x^3 + 3} = 0$

ii). $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is infinite if the degrees of $f(x) >$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{x^3 + 2x}{x^2 - 8} = \infty$

iii). $\lim_{x \to \pm\infty} \frac{f(x)}{g(x)}$ is finite if the degree of $f(x) =$ the degree of $g(x)$

Example: $\lim_{x \to \infty} \frac{2x^2 - 3x + 2}{10x - 5x^2} = -\frac{2}{5}$

L'Hôpital's Rule

If $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and if $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then

$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

Intermediate-Value Theorem

A function $y = f(x)$ that is continuous on a closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$.

Note: If f is continuous on $[a,b]$ and $f(a)$ and $f(b)$ differ in sign, then the equation $f(x) = 0$ has at least one solution in the open interval (a,b).

Horizontal and Vertical Asymptotes

1. A line $y = b$ is a horizontal asymptote of the graph $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \text{ or } \lim_{x \to -\infty} f(x) = b.$$

2. A line $x = a$ is a vertical asymptote of the graph $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \text{ or } \lim_{x \to a^-} = \pm\infty.$$

Average and Instantaneous Rate of Change

i). *Average Rate of Change*: If (x_0, y_0) and (x_1, y_1) are points on the graph of $y = f(x)$, then the average rate of change of y with respect to x over the interval $[x_0, x_1]$ is $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{\Delta y}{\Delta x}$.

ii). *Instantaneous Rate of Change*: If (x_0, y_0) is a point on the graph of $y = f(x)$, then the instantaneous rate of change of y with respect to x at x_0 is $f'(x_0)$.

Definition of Derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{of } f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

The latter definition of the derivative is the instantaneous rate of change of $f(x)$ with respect to x at x = a.

Mean Value Theorem

If f is continuous on $[a,b]$ and differentiable on (a,b), then there is at least one number c in (a,b) such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

Extreme-Value Theorem

If f is continuous on a closed interval $[a,b]$, then $f(x)$ has both a maximum and minimum on $[a,b]$.

To find the maximum and minimum values of a function $y = f(x)$, locate

1. the points where $f'(x)$ is zero or where $f'(x)$ fails to exist.
2. the end points, if any, on the domain of $f(x)$.

Note: These are the only candidates for the value of x where $f(x)$ may have a maximum or a minimum.

Let f be differentiable for $a < x < b$ and continuous for a $a \le x \le b$,

1. If $f'(x) > 0$ for every x in (a,b), then f is increasing on $[a,b]$.
2. If $f'(x) < 0$ for every x in (a,b), then f is decreasing on $[a,b]$.

Suppose that $f''(x)$ exists on the interval (a,b)

1. If $f''(x) > 0$ in (a,b), then f is concave upward in (a,b).
2. If $f''(x) < 0$ in (a,b), then f is concave downward in (a,b).

To locate the points of inflection of $y = f(x)$, find the points where $f''(x) = 0$ <u>or</u> where $f''(x)$ fails to exist. These are the only candidates where $f(x)$ may have a point of inflection. Then test these points to make sure that $f''(x) < 0$ on one side and $f''(x) > 0$ on the other.

If a function is differentiable at point $x = a$, it is continuous at that point. The converse is false, in other words, continuity does <u>not</u> imply differentiability.

<u>Local Linearity and Linear Approximations</u>

The linear approximation to $f(x)$ near $x = x_0$ is given by $y = f(x_0) + f'(x_0)(x - x_0)$ for x sufficiently close to x_0.

To estimate the slope of a graph at a point – just draw a tangent line to the graph at that point. Another way is (by using a graphing calculator) to "zoom in" around the point in question until the graph "looks" straight. This method almost always works. If we "zoom in" and the graph looks straight at a point, say $(a, f(a))$, then the function is <u>locally linear</u> at that point.

The graph of $y = |x|$ has a sharp corner at $x = 0$. This corner cannot be smoothed out by "zooming in" repeatedly. Consequently, the derivative of $|x|$ does not exist at $x = 0$, hence, is <u>not</u> locally linear at $x = 0$.

If f is differentiable at every point on an interval I, and $f'(x) \neq 0$ on I, then $g = f^{-1}(x)$ is differentiable at every point of the interior of the interval $f(I)$ and

$$g'(f(x)) = \frac{1}{f'(x)}.$$

Properties of $y = e^x$

1. The exponential function $y = e^x$ is the inverse function of $y = \ln x$.
2. The domain is the set of all real numbers, $-\infty < x < \infty$.
3. The range is the set of all positive numbers, $y > 0$.
4. $\frac{d}{dx}(e^x) = e^x$
5. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
6. $y = e^x$ is continuous, increasing, and concave up for all x.
7. $\lim_{x \to +\infty} e^x = +\infty$ and $\lim_{x \to -\infty} e^x = 0$.
8. $e^{\ln x} = x$, for $x > 0$; $\ln(e^x) = x$ for all x.

Properties of $y = \ln x$

1. The domain of $y = \ln x$ is the set of all positive numbers, $x > 0$.
2. The range of $y = \ln x$ is the set of all real numbers, $-\infty < y < \infty$.
3. $y = \ln x$ is continuous and increasing everywhere on its domain.
4. $\ln(ab) = \ln a + \ln b$.
5. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.
6. $\ln a^r = r \ln a$.
7. $y = \ln x < 0$ if $0 < x < 1$.
8. $\lim_{x \to +\infty} \ln x = +\infty$ and $\lim_{x \to 0^+} \ln x = -\infty$.
9. $\log_a x = \frac{\ln x}{\ln a}$

Definition of Definite Integral as the Limit of a Sum

Suppose that a function $f(x)$ is continuous on the closed interval $[a, b]$. Divide the interval into n equal subintervals, of length $\Delta x = \frac{b-a}{n}$. Choose one number in each subinterval, in other words, x_1 in the first, x_2 in the second, ..., x_k in the k th ,..., and x_n in the n th . Then

$$\lim_{n \to \infty} \sum_{k=1}^{n} f(x_k)\Delta x = \int_a^b f(x)\, dx = F(b) - F(a).$$

Properties of the Definite Integral

Let $f(x)$ and $g(x)$ be continuous on $[a,b]$.

i). $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$ for any constant c.

ii). $\int_a^a f(x)\,dx = 0$

iii). $\int_a^b \mathrm{f(x)\,dx} = -\int_b^a \mathrm{f(x)\,dx}$

iv). $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where f is continuous on an interval containing the numbers a, b, and c.

Fundamental Theorem of Calculus:

(has two parts)

1.

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F'(x) = f(x), \text{ or } \frac{d}{dx}\int_a^b f(x)\,dx = f(x).$$

2.

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \quad \text{or} \quad \frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(x)\cdot g'(x)$$

Velocity, Speed, and Acceleration

1. The velocity of an object tells how fast it is going and in which direction. Velocity is an instantaneous rate of change.
2. The speed of an object is the absolute value of the velocity, $|v(t)|$. It tells how fast it is going disregarding its direction.
   The speed of a particle increases (speeds up) when the velocity and acceleration have the same signs. The speed decreases (slows down) when the velocity and acceleration have opposite signs.
3. The acceleration is the instantaneous rate of change of velocity – it is the derivative of the velocity – that is, $a(t) = v'(t)$. Negative acceleration (deceleration) means that the velocity is decreasing. The acceleration gives the rate ot which the velocity is changing.

Therefore, if x is the displacement of a moving object and t is time, then:

i) velocity $= v(t) = x'(t) = \frac{dx}{dt}$

ii) acceleration $= a(t) = x''(t) = v'(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

iii) $v(t) = \int a(t)\,dt$

iv) $x(t) = \int v(t)\,dt$

Note: The average velocity of a particle over the time interval from t_0 to another time t, is

Average Velocity $= \frac{\text{Change in position}}{\text{Length of time}} = \frac{s(t) - s(t_0)}{t - t_0}$, where $s(t)$ is the position of the particle at time t.

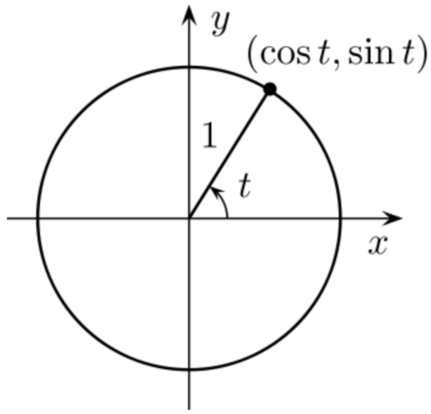
Solving Differential Equations: Graphically and Numerically

Slope Fields

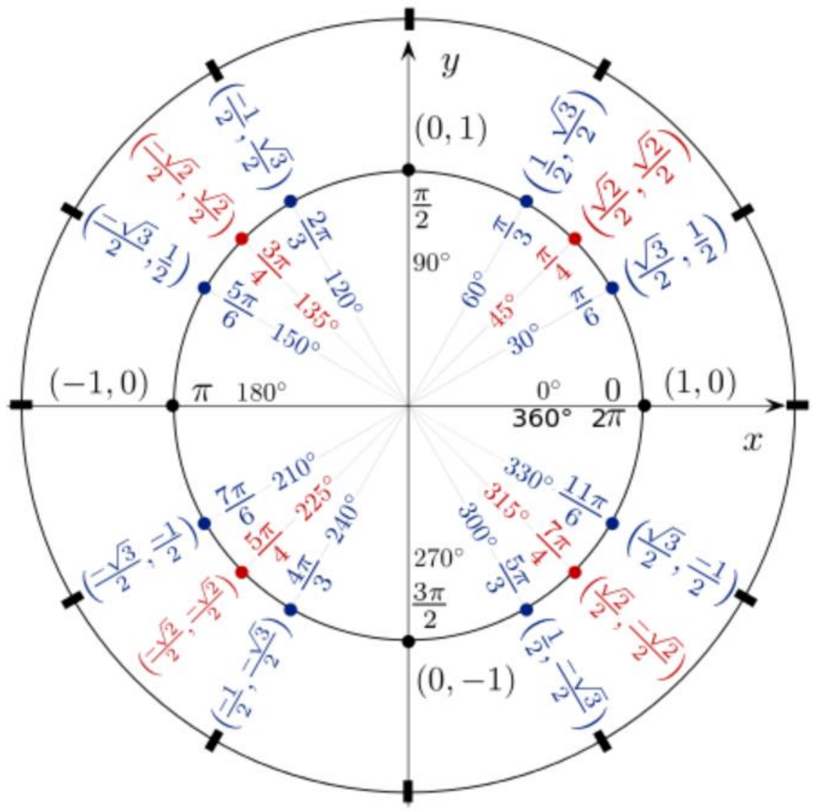
At every point (x, y) a differential equation of the form $\frac{dy}{dx} = f(x, y)$ gives the slope of the member of the family of solutions that contains that point. A slope field is a graphical representation of this family of curves. At each point in the plane, a short segment is drawn whose slope is equal to the value of the derivative at that point. These segments are tangent to the solution's graph at the point.

The slope field allows you to sketch the graph of the solution curve even though you do not have its equation. This is done by starting at any point (usually the point given by the initial condition), and moving from one point to the next in the direction indicated by the segments of the slope field.

Trig Review



(Remember that for AP Calc we only use the radian measures of the angles.)



$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$
$$\sec(\theta) = \frac{1}{\cos(\theta)}$$
$$\csc(\theta) = \frac{1}{\sin(\theta)}$$
$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

$$\sin^2\theta + \cos^2\theta = 1$$
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$